

PERMUTATION GROUPS AND DERANGEMENTS OF ODD PRIME ORDER

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ABSTRACT. Let G be a transitive permutation group of degree n . We say that G is $2'$ -elusive if n is divisible by an odd prime, but G does not contain a derangement of odd prime order. In this paper we study the structure of quasiprimitive and biquasiprimitive $2'$ -elusive permutation groups, extending earlier work of Giudici and Xu on elusive groups. As an application, we use our results to investigate automorphisms of finite arc-transitive graphs of prime valency.

1. INTRODUCTION

Let $G \leq \text{Sym}(\Omega)$ be a transitive permutation group on a finite set Ω of size at least 2. An element $x \in G$ is a *derangement* if it acts fixed-point-freely on Ω . Equivalently, if H is a point stabiliser, then x is a derangement if and only if the conjugacy class of x fails to meet H . An easy application of the Orbit-Counting Lemma shows that G contains derangements. This classical theorem of Jordan has interesting applications in number theory and topology (see Serre's article [24], for example).

By a theorem of Fein, Kantor and Schacher [11], G contains a derangement of prime power order. This result turns out to have some important number-theoretic applications; for example, it implies that the relative Brauer group of any nontrivial extension of global fields is infinite (see [11, Corollary 4]). It is worth noting that the existence of a derangement of prime power order in [11] requires the Classification of Finite Simple Groups. In most cases, G contains a derangement of prime order, but there are some exceptions, such as the 3-transitive action of the smallest Mathieu group M_{11} on 12 points. The transitive permutation groups with this property are called *elusive* groups, and they have been the subject of many papers in recent years; see [6, 12, 13, 14, 15, 26], for example.

A local notion of elusivity was introduced in [5]. Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group and let r be a prime divisor of $|\Omega|$. We say that G is r -*elusive* if it does not contain a derangement of order r (so G is elusive if and only if it is r -elusive for every prime divisor r of $|\Omega|$). In [5], all the r -elusive primitive almost simple groups with socle an alternating or sporadic group are determined. This work has been extended in our recent book [4], which provides a detailed study of r -elusive classical groups. The r -elusive notion leads naturally to the definition of a $2'$ -elusive permutation group, which are the main focus of this paper. We say that a finite transitive permutation group $G \leq \text{Sym}(\Omega)$ is $2'$ -*elusive* if $|\Omega|$ is divisible by an odd prime, but G does not contain a derangement of odd prime order.

Let $G \leq \text{Sym}(\Omega)$ be a transitive permutation group with point stabiliser H . Recall that G is *primitive* if H is a maximal subgroup of G , and note that every nontrivial normal subgroup of a primitive group is transitive. This observation suggests a natural generalisation of primitivity; we say that G is *quasiprimitive* if every nontrivial normal subgroup is transitive. Similarly, G is *biquasiprimitive* if every nontrivial normal subgroup has at most two orbits on Ω , and there is at least one nontrivial normal subgroup with two orbits.

Quasiprimitive and biquasiprimitive groups arise naturally in the study of finite vertex-transitive graphs. For example, if G is a vertex-transitive group of automorphisms of a graph Γ such that for each vertex v , the action of the vertex stabiliser G_v on the set of

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neighbours of v is quasiprimitive (that is, Γ is G -locally-quasiprimitive), then [21, Lemma 1.6] implies that every normal subgroup N of G with at least three orbits is semiregular (that is, $N_v = 1$ for every vertex v). In this situation, the quotient graph with respect to the orbits of such a normal subgroup inherits many of the symmetry properties of the original graph Γ . This explains why quasiprimitive and biquasiprimitive groups often arise as base cases in the analysis of various families of vertex-transitive graphs, see for example [9, 22]. These important graph-theoretic applications motivated Praeger to establish detailed structure theorems for quasiprimitive [22] and biquasiprimitive groups [23]. The structure theorem for quasiprimitive groups is similar to the celebrated O’Nan-Scott Theorem for primitive groups.

The elusive quasiprimitive permutation groups have been determined by Giudici (see [12, Theorem 1.1]); the only examples are primitive groups of the form $G = M_{11} \text{ wr } K$ in its product action on $\Omega = \Delta^k$, where $K \leq S_k$ is transitive and $|\Delta| = 12$. Further progress has been made by Giudici and Xu in [15], where the biquasiprimitive elusive groups are determined (see [15, Theorem 1.4]). As an application, they prove that every finite vertex-transitive, locally-quasiprimitive graph Γ has a semiregular automorphism (in other words, the automorphism group $\text{Aut}(\Gamma)$, viewed as a permutation group on the set of vertices of Γ , contains a derangement of prime order); see [15, Theorem 1.1]. This result settles an important case of the *Polycirculant Conjecture* from 1981, which asserts that every finite vertex-transitive digraph has a semiregular automorphism [6, 17]. For example, [15, Theorem 1.1] immediately implies that the conjecture holds for every finite arc-transitive graph of prime valency.

The main goal of this paper is to extend this earlier work from elusive to $2'$ -elusive groups. We begin by determining the primitive $2'$ -elusive groups.

Theorem 1. *Let $G \leq \text{Sym}(\Omega)$ be a finite primitive permutation group. Then G is $2'$ -elusive if and only if*

$$\text{soc}(L)^k \trianglelefteq G \leq L \text{ wr } K$$

and G acts with its product action on $\Omega = \Delta^k$ for some $k \geq 1$, where $L \leq \text{Sym}(\Delta)$ is almost simple and primitive with stabiliser J , G induces the transitive subgroup $K \leq S_k$ on the set of k simple direct factors of $\text{soc}(G) = \text{soc}(L)^k$ and one of the following holds:

- (i) $L = M_{11}$ and $J = \text{PSL}_2(11)$;
- (ii) $L = {}^2F_4(2)$ and $J = \text{PSL}_2(25).2_3$.

Remark. In both cases, 3 is the only odd prime divisor of $|\Omega|$. In case (i), $G = M_{11} \text{ wr } K$, $G_\alpha = \text{PSL}_2(11) \text{ wr } K$ and G is elusive. In (ii), $\text{PSL}_2(25).2_3$ is an almost simple nonsplit extension.

Our next result describes the structure of the quasiprimitive $2'$ -elusive groups (in view of Theorem 1, we may assume that G is imprimitive). In the statement, $\text{soc}(L)_\delta$ denotes the stabiliser in $\text{soc}(L)$ of a point $\delta \in \Delta$.

Theorem 2. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive quasiprimitive imprimitive permutation group with point stabiliser H . Then G acts faithfully on a nontrivial system of imprimitivity \mathcal{P} for Ω , which can be identified with a Cartesian product Δ^k for some $k \geq 2$, and*

$$\text{soc}(L)^k \trianglelefteq G \leq L \text{ wr } K$$

where G induces the transitive subgroup $K \leq S_k$ on the set of k simple direct factors of $\text{soc}(G) = \text{soc}(L)^k$. Moreover, $L \leq \text{Sym}(\Delta)$ is transitive with $\text{soc}(L) = \text{PSL}_2(p)$ for some Mersenne prime p , $\text{soc}(L)_\delta = C_p : C_{(p-1)/2}$ and

$$(C_p : C_r)^k \leq H \cap \text{soc}(L)^k < (C_p : C_{(p-1)/2})^k$$

where r is the product of the distinct prime divisors of $(p-1)/2$.

In order to state our final result, recall that a transitive group $G \leq \text{Sym}(\Omega)$ is biquasiprimitive if every nontrivial normal subgroup of G has at most two orbits on Ω , and there is a normal subgroup with two orbits, say Δ_1 and Δ_2 . Let G^+ denote the index-two subgroup of G that fixes Δ_1 and Δ_2 setwise.

Theorem 3. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser $H = G_\alpha$ and minimal normal subgroup N . Let $K \leq S_k$ be the transitive group induced by G on the set of k simple direct factors of N and let $K^+ \leq K$ be the group induced by G^+ .*

- (a) $G^+ = NH$ and N is the unique minimal normal subgroup of G .
- (b) If G^+ acts faithfully on its two orbits, then one of the following holds:
 - (i) $(G, H) = (M_{10}, A_5)$ or $(\text{Aut}(A_6), S_5)$;
 - (ii) $G = M_{11} \text{ wr } K$, $H = \text{PSL}_2(11) \text{ wr } K^+$ and $|K : K^+| = 2$;
 - (iii) $N = ({}^2F_4(2)')^k \trianglelefteq G \leq {}^2F_4(2) \text{ wr } K$ and $H = N_{G^+}(N_\alpha)$, where $N_\alpha = \text{PSL}_2(25)^k$;
 - (iv) $N = \text{PSL}_2(p)^k \trianglelefteq G \leq \text{PGL}_2(p) \text{ wr } K$,

$$(C_p : C_r)^k \leq N_\alpha < (C_p : C_{(p-1)/2})^k$$

and

$$H < N_{G^+}(N_\alpha) = G^+ \cap ((C_p : C_{p-1}) \text{ wr } K),$$

where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Moreover, each group G in (i), (ii) and (iii) is $2'$ -elusive and biquasiprimitive.

- (c) If G^+ is not faithful on at least one orbit, then k is even, K^+ is intransitive, $|K : K^+| = 2$ and one of the following holds:
 - (i) $G = M_{11} \text{ wr } K$ and $H = (\text{PSL}_2(11)^{k/2} \times M_{11}^{k/2}) : K^+$;
 - (ii) $N = ({}^2F_4(2)')^k \trianglelefteq G \leq {}^2F_4(2) \text{ wr } K$, $N_\alpha = \text{PSL}_2(25)^{k/2} \times ({}^2F_4(2)')^{k/2}$ and

$$H = N_{G^+}(N_\alpha) = G^+ \cap ((\text{PSL}_2(25).2_3)^{k/2} \times {}^2F_4(2)^{k/2}) : K^+$$
 with $G^+ = G \cap ({}^2F_4(2) \text{ wr } K^+)$;
 - (iii) $N = \text{PSL}_2(p)^k \trianglelefteq G \leq \text{PGL}_2(p) \text{ wr } K$,

$$(C_p : C_r)^{k/2} \times \text{PSL}_2(p)^{k/2} \leq N_\alpha < (C_p : C_{(p-1)/2})^{k/2} \times \text{PSL}_2(p)^{k/2}$$

and

$$H < N_{G^+}(N_\alpha) = G^+ \cap ((C_p : C_{p-1})^{k/2} \times \text{PGL}_2(p)^{k/2}) : K^+,$$

where $G^+ = G \cap (\text{PGL}_2(p) \text{ wr } K^+)$ and p is a Mersenne prime.

Moreover, each group G in (i) and (ii) is $2'$ -elusive and biquasiprimitive.

Corollary 4. *Let $G \leq \text{Sym}(\Omega)$ be a finite quasiprimitive or biquasiprimitive permutation group such that $|\Omega|$ is divisible by a prime $q \geq 5$. Then either G contains a derangement of odd prime order, or $\text{PSL}_2(p)^k$ is the unique minimal normal subgroup of G , where $k \geq 1$ and p is a Mersenne prime such that q^2 divides $(p-1)/2$.*

Remark. Referring to Corollary 4, it is worth noting that $2^{61} - 1$ is the smallest Mersenne prime p with the property that $(p-1)/2$ is divisible by q^2 for a prime $q \geq 5$.

Recall that the Polycirculant Conjecture asserts that every finite vertex-transitive graph has a semiregular automorphism. The existence of such an automorphism has numerous applications. For instance, they have been used to construct Hamiltonian paths and cycles [1], to provide succinct representations of graphs [2], and to enumerate all vertex-transitive graphs of small orders [19]. In many of these applications, it is desirable to work with a semiregular automorphism of order as large as possible. For example, the Polycirculant Conjecture is established for all vertex-transitive cubic graphs in [18], and later work of

Cameron et al. [7] shows that any such graph admits a semiregular automorphism of order greater than two. In fact, the main theorem of [20] reveals that there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$, satisfying $f(n) \rightarrow \infty$ as $n \rightarrow \infty$, such that any vertex-transitive cubic graph on n vertices contains a semiregular subgroup of order at least $f(n)$.

As noted above, Giudici and Xu use their work on elusive groups in [15] to verify the Polycirculant Conjecture for all finite arc-transitive graphs of prime valency p . We anticipate that our results on $2'$ -elusive groups in this paper will play a key role in establishing the conjecture for arc-transitive graphs of valency $2p$. A key tool in order to achieve this goal is Theorem 5 below, which may be of independent interest (the application to graphs of valency $2p$ will be the subject of a future paper). This approach is similar to the aforementioned extension of the main theorem of [18] in [7].

In the statement of the theorem, $V\Gamma$ denotes the set of vertices of Γ , and K_{12} is the complete graph on 12 vertices. In addition, the *standard double cover* of Γ is the graph with vertex set $V\Gamma \times \{0, 1\}$, such that $\{(u, a), (v, b)\}$ is an edge if and only if $a \neq b$ and $\{u, v\}$ is an edge of Γ .

Theorem 5. *Let Γ be a finite connected graph of prime valency p and let $G \leq \text{Aut}(\Gamma)$ be an arc-transitive group of automorphisms so that the action of G on $V\Gamma$ is either quasiprimitive or biquasiprimitive. Then one of the following holds:*

- (i) G contains a derangement of odd prime order;
- (ii) $|V\Gamma|$ is a power of 2;
- (iii) $\Gamma = K_{12}$, $G = M_{11}$ and $p = 11$;
- (iv) $|V\Gamma| = (p^2 - 1)/2s$ and $G = \text{PSL}_2(p)$ or $\text{PGL}_2(p)$, where p is a Mersenne prime and $C_r \leq C_s < C_{(p-1)/2}$, where r is the product of the distinct prime divisors of $(p-1)/2$;
- (v) $|V\Gamma| = (p^2 - 1)/s$ and $G = \text{PGL}_2(p)$, where p and s are as in part (iv), and Γ is the standard double cover of the graph given in (iv).

As we will explain in Section 5, if $G \leq \text{Sym}(\Omega)$ is a finite transitive permutation group then there is a one-to-one correspondence between the set of suborbits of G and the set of finite digraphs with vertex set Ω on which G acts arc-transitively. Moreover, the connected graphs of valency p correspond to self-paired suborbits ω^{G_α} of length p with the property that $G = \langle G_\alpha, g \rangle$ for each $g \in G$ that interchanges α and ω . Therefore, one of the main steps in the proof of Theorem 5 is to determine the $2'$ -elusive quasiprimitive and biquasiprimitive groups with a prime subdegree; we can do this by applying Theorems 1, 2 and 3. In the cases that arise, we then need to check that G has a suborbit with the appropriate properties.

Finally, we record a couple of corollaries to Theorem 5 (the short proofs are presented at the end of Section 5).

Corollary 6. *Let Γ be a finite connected graph of prime valency and let $G \leq \text{Aut}(\Gamma)$ be an elusive arc-transitive group of automorphisms. Then $\Gamma = K_{12}$ and $G = M_{11}$.*

Corollary 7. *The smallest integer k such that there is a finite connected graph of valency k with an elusive arc-transitive group of automorphisms is 6.*

Note that Corollary 7 answers a question posed in [14]. The analogous question for vertex-transitive groups remains open.

Notation. Our notation is standard. We write $H.K$ to denote an extension of H by K , and $H:K$ if the extension splits. If n is a positive integer then C_n denotes a cyclic group of order n , and H^n is the direct product of n copies of H . If p is a prime, then $O_p(H)$ denotes the largest normal p -subgroup of H . Finally, if H acts on a set Δ then we write H^Δ to denote the induced permutation group on Δ .

T	H
$\text{PSL}_2(p)$	$C_p:C_r \leq H \leq C_p:C_{(p-1)/2}$ (see Remark 2.1)
$\text{P}\Omega_8^+(3)$	$\Omega_7(3)$
$\Omega_8^+(2)$	$\text{Sp}_6(2)$
$\Omega_8^+(2)$	A_9
${}^2F_4(2)'$	$\text{PSL}_2(25)$
A_6	A_5
M_{11}	$\text{PSL}_2(11)$

TABLE 1. The cases (T, H) in Theorem 2.2(i)

2. SIMPLE GROUPS

In [12, Theorem 1.3], Giudici determines the nonabelian finite simple groups T with a proper subgroup that meets every $\text{Aut}(T)$ -conjugacy class of elements of prime order. We can adopt a similar approach in order to establish an analogous result for odd primes, which will play a key role in the proofs of our main theorems.

Remark 2.1. In the first row of Table 1, p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$. Also observe that $|T : H|$ is a 2-power if $H = C_p:C_{(p-1)/2}$.

Theorem 2.2. *Let T be a nonabelian finite simple group.*

- (i) *T has a proper subgroup H that meets every $\text{Aut}(T)$ -class of elements of odd prime order in T if and only if (T, H) is one of the cases in Table 1.*
- (ii) *In addition, H meets every T -class of elements of odd prime order in T if and only if $T = {}^2F_4(2)'$, M_{11} or $\text{PSL}_2(p)$ with p a Mersenne prime.*

Proof. Suppose $H < T$ is a proper subgroup that meets every $\text{Aut}(T)$ -class of elements of odd prime order in T , so every odd prime divisor of $|T|$ also divides $|H|$. Moreover, if $H \leq K \leq T$ then every $\text{Aut}(T)$ -class of elements of odd prime order in T meets K . Thus we will initially assume that H is a maximal subgroup of T ; if (T, H) is an example then we will need to check if any proper subgroups of H also meet every $\text{Aut}(T)$ -class of elements of odd prime order.

First assume that T is a sporadic simple group. Here the possibilities for T and H (with H maximal and $|H|$ divisible by every odd prime divisor of $|T|$) can be read off from [16, Table 10.6]:

$(M_{11}, \text{PSL}_2(11))$	$(M_{12}, \text{PSL}_2(11))$	(M_{12}, M_{11})	(M_{24}, M_{23})
(HS, M_{22})	(McL, M_{22})	(Co_2, M_{23})	(Co_3, M_{23})

It follows that $\pi(H) = \pi(T)$, where $\pi(X)$ is the set of prime divisors of $|X|$. The cases (McL, M_{22}) and (Co_2, M_{23}) are ruled out in [12, Section 3.11], where an $\text{Aut}(T)$ -class of elements of odd prime order not meeting H is identified. If $T = M_{11}$, M_{24} or Co_3 then $T = \text{Aut}(T)$ and by applying [5, Corollary 1.2] we deduce that $(M_{11}, \text{PSL}_2(11))$ is the only example. In addition, no proper subgroup of $\text{PSL}_2(11)$ has order divisible by every odd prime divisor of $|M_{11}|$, so no further examples arise. In the remaining three cases, we can use [8] to identify an $\text{Aut}(T)$ -class of elements of odd prime order r that does not meet H (indeed, take $r = 5$ if $(T, H) = (\text{HS}, M_{22})$, and $r = 3$ in the other two cases).

Next assume $T = A_n$ is an alternating group. Since the two largest primes at most n must divide $|H|$, [16, Theorem 4] implies that $H \cong (S_k \times S_{n-k}) \cap T$ for some $1 \leq k < n/2$ (note that this includes the case $(T, H) = (A_6, \text{PSL}_2(5))$). In particular, the action of T on the set of right cosets of H is permutation isomorphic to the action of T on the set of subsets of $\{1, \dots, n\}$ of size k . Since this action extends to S_n , it follows that if $n \neq 6$ then the $\text{Aut}(T)$ -class of an element $t \in T$ meets H if and only if t fixes a k -set. By a theorem of

G	H
$\text{PSL}_2(p)$ or $\text{PGL}_2(p)$	$C_p:C_r \leq H < C_p:C_{(p-1)/2}$
$\text{PGL}_2(p)$	$C_p:C_{2r} \leq H < C_p:C_{p-1}$
${}^2F_4(2)'$ or ${}^2F_4(2)$	$\text{PSL}_2(25)$
${}^2F_4(2)$	$\text{PSL}_2(25).2_3$
M_{10} or $\text{Aut}(A_6)$	A_5
$\text{Aut}(A_6)$	S_5
M_{11}	$\text{PSL}_2(11)$

TABLE 2. The $2'$ -elusive almost simple groups

Sylvester [25], $\binom{n}{k}$ is divisible by an odd prime, so [5, Corollary 3.2(iii)] implies that there is an $\text{Aut}(T)$ -class of elements of odd prime order that does not meet H . Finally, if $n = 6$ then it is easy to check that the only subgroups H of T with the required property are isomorphic to A_5 . In addition, note that A_5 has a unique class of elements of order 3, but A_6 has two, so H does not meet every T -class of elements of odd prime order.

For the remainder, we may assume that T is a simple group of Lie type. By [16, Theorem 4(i)], the possibilities for T and H can be read off from [16, Tables 10.1–10.5]. More precisely, these tables give the proper subgroups M of T with the property that $|M|$ is divisible by a specific collection of odd prime divisors of $|T|$. By inspection, and recalling that we are assuming H is maximal, we deduce that either $|H|$ is even, or $(T, H) = (\text{PSL}_3(3), C_{13}:C_3)$ or $(\text{PSL}_2(p), C_p:C_{(p-1)/2})$ for a Mersenne prime p .

If $(T, H) = (\text{PSL}_3(3), C_{13}:C_3)$ then T has two $\text{Aut}(T)$ -classes of subgroups of order 3, and H has a unique such class, so there is an $\text{Aut}(T)$ -class of elements of order 3 that misses H . Now assume $(T, H) = (\text{PSL}_2(p), C_p:C_{(p-1)/2})$ with p a Mersenne prime. Here T has a unique class of subgroups of each odd prime order, hence every T -class of elements of odd prime order meets H . The same conclusion holds for any subgroup $L < H$ with $\pi(H) = \pi(L)$, so we deduce that

$$C_p:C_r \leq H \leq C_p:C_{(p-1)/2}$$

as in the first row of Table 1 (where r is the product of the distinct prime divisors of $(p-1)/2$).

To complete the proof we may assume that $|H|$ is even and thus $\pi(T) = \pi(H)$. Here the possibilities for (T, H) can be read off from [16, Table 10.7]. These cases were studied in [12, Section 3], where in most instances an $\text{Aut}(T)$ -class of elements of odd prime order that misses H is identified. The exceptions are as follows:

$$\begin{array}{lll} (\text{P}\Omega_8^+(3), \Omega_7(3)) & (\Omega_8^+(2), A_9) & (\Omega_8^+(2), \text{Sp}_6(2)) \\ (\text{PSU}_6(2), M_{22}) & (\text{PSU}_5(2), \text{PSL}_2(11)) & ({}^2F_4(2)', \text{PSL}_2(25)) \end{array}$$

For the cases with $T = \text{PSU}_6(2)$ or $\text{PSU}_5(2)$, one can use MAGMA [3], or the information in [8], to check that there is an $\text{Aut}(T)$ -class of elements of order 3 that misses H . The remaining four cases are recorded in Table 1. In each of these cases it is easy to see that H does not contain a proper subgroup with the desired property, so we do not obtain any additional examples. Finally, as explained in [12, Sections 3.8–3.9], if (T, H) is one of the examples in Table 1 with $T = \Omega_8^+(2)$ or $\text{P}\Omega_8^+(3)$ then H does not meet every T -class of elements of odd prime order, so (T, H) does not arise in part (ii) of Theorem 2.2. \square

By applying Theorem 2.2, we can determine all the $2'$ -elusive almost simple groups. In Table 2, as before, p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Theorem 2.3. *Let $G \leq \text{Sym}(\Omega)$ be a finite transitive almost simple permutation group with point stabiliser H . Then G is $2'$ -elusive if and only if (G, H) is one of the cases in Table 2.*

Proof. Let T denote the socle of G and assume that G is $2'$ -elusive. Then $H \cap T$ meets every G -class of elements in T of odd prime order, so $(T, H \cap T)$ is one of the cases arising in Theorem 2.2(i).

First assume T is a transitive subgroup of G . Here $G = TH$ and thus T is $2'$ -elusive since H meets every T -class of elements of odd prime order in T . Note that $|G : T| = |H : H \cap T|$. By applying Theorem 2.2(ii) we deduce that (G, H) is one of the following:

- (a) $G = M_{11}$, $H = \text{PSL}_2(11)$
- (b) $G = {}^2F_4(2)'$, $H = \text{PSL}_2(25)$
- (c) $G = {}^2F_4(2)$, $H = \text{PSL}_2(25).2_3$
- (d) $G = \text{PSL}_2(p)$, $C_p : C_r \leq H < C_p : C_{(p-1)/2}$
- (e) $G = \text{PGL}_2(p)$, $C_p : C_{2r} \leq H < C_p : C_{p-1}$

where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$. Note that in case (d) (and similarly in (e)) we require $H < C_p : C_{(p-1)/2}$ since $|\Omega|$ is divisible by an odd prime. Also observe that G is primitive in cases (a), (b) and (c), and quasiprimitive (and imprimitive) in cases (d) and (e).

Now assume T is intransitive, in which case the orbits of T on Ω have equal size and the actions of T on each orbit are isomorphic. Clearly, $T \neq M_{11}$. If $T = {}^2F_4(2)'$ or $\text{PSL}_2(p)$ (with p a Mersenne prime) then (G, H) is one of the following:

- (f) $G = {}^2F_4(2)$, $H = \text{PSL}_2(25)$
- (g) $G = \text{PGL}_2(p)$, $C_p : C_r \leq H < C_p : C_{(p-1)/2}$

where r is the product of the distinct prime divisors of $(p-1)/2$ as before. Next suppose $T = \Omega_8^+(2)$ or $\text{P}\Omega_8^+(3)$. As explained in [12, Section 4], G must contain a triality graph automorphism (if not, there are derangements of order 5), but this implies that G contains an element of order 3 that permutes the orbits of T , which is a derangement. Finally, let us assume $T = A_6$, so $G \in \{S_6, M_{10}, \text{PGL}_2(9), \text{Aut}(A_6)\}$. Here G is $2'$ -elusive if and only if the two T -classes of elements of order 3 are fused in G , so (G, H) is one of the following:

- (h) $G = M_{10}$, $H = A_5$
- (i) $G = \text{Aut}(A_6)$, $H \in \{A_5, S_5\}$

This completes the proof of Theorem 2.3. \square

Corollary 2.4. *Let $G \leq \text{Sym}(\Omega)$ be a finite primitive almost simple permutation group with point stabiliser H . Then G is $2'$ -elusive if and only if*

$$(G, H) = (M_{11}, \text{PSL}_2(11)), ({}^2F_4(2)', \text{PSL}_2(25)) \text{ or } ({}^2F_4(2), \text{PSL}_2(25).2_3).$$

3. QUASIPRIMITIVE GROUPS

In this section we investigate the structure of $2'$ -elusive quasiprimitive groups. Our aim is to prove Theorems 1 and 2. We begin by recording a standard lemma which will be useful later.

Lemma 3.1. *Let $G \leq \text{Sym}(\Omega)$ be a finite permutation group with a transitive normal subgroup $N = T^k$ such that $C_G(N) = 1$, where T is a nonabelian simple group and $k \geq 2$. Let $\alpha \in \Omega$ and assume that $N_\alpha = S^k$ for some proper subgroup $S < T$. Then we can identify Ω with the Cartesian product Δ^k , where $\Delta = T/S$, such that G is permutation isomorphic to a subgroup of $\text{Aut}(T) \text{ wr } S_k$ acting on Δ^k with its usual product action.*

Proof. First observe that G is isomorphic to a subgroup L of $\text{Aut}(N) = \text{Aut}(T) \text{ wr } S_k$ since $C_G(N) = 1$. Since N acts transitively on Ω we have $G = NG_\alpha$. Let Σ be the set of right cosets of N_α in N .

Define an action of G on Σ by $(N_\alpha m)^{ng} = N_\alpha (mn)^g$ for each $m, n \in N$ and $g \in G_\alpha$. To see that this is well-defined, first observe that if $N_\alpha m_1 = N_\alpha m_2$ then $m_1 m_2^{-1} \in N_\alpha$. Therefore,

since $N_\alpha \triangleleft G_\alpha$, we deduce that

$$(m_1 n)^g ((m_2 n)^g)^{-1} = m_1^g (m_2^{-1})^g = (m_1 m_2^{-1})^g \in N_\alpha$$

and thus $(N_\alpha m_1)^{ng} = (N_\alpha m_2)^{ng}$. In addition, if $n_1 g_1 = n_2 g_2$ then $n_2^{-1} n_1 = g_2 g_1^{-1} \in N_\alpha$ and it follows that $g_2^{-1} g_1 \in N_\alpha$ since $N_\alpha \triangleleft G_\alpha$. A routine calculation now shows that $(N_\alpha m)^{n_1 g_1} = (N_\alpha m)^{n_2 g_2}$ and so the action of G on Σ is well-defined. Since the stabiliser in G of the trivial coset $N_\alpha \in \Sigma$ is G_α , it follows that the action of G on Ω is permutation isomorphic to the action of G on Σ .

Let $\varphi : G \rightarrow L$ be the isomorphism induced by the conjugation action of G on N . Let Λ be the set of right cosets of S^* in T^* , where $T^* = \text{Inn}(T)$ and S^* is the group of automorphisms of T induced by conjugation by elements of S . Let $\rho : \Sigma \rightarrow \Lambda^k$ be the bijection sending $N_\alpha(t_1, \dots, t_k)$ to $(S^* t_1^*, \dots, S^* t_k^*)$, where t_i^* is the inner automorphism of T induced by conjugation by the element $t_i \in T$. Now L acts on Λ^k via the product action: if $x = ng \in G$, with $n \in N$ and $g \in G_\alpha$, then $\varphi(n) = (a_1, \dots, a_k) \in \text{Inn}(T)^k$ with

$$(S^* t_1^*, \dots, S^* t_k^*)^{\varphi(n)} = (S^* t_1^* a_1, \dots, S^* t_k^* a_k)$$

and $\varphi(g) = (b_1, \dots, b_k)\pi \in \text{Aut}(T) \text{ wr } S_k$ with

$$(S^* t_1^*, \dots, S^* t_k^*)^{\varphi(g)} = (S^* (t_{1\pi^{-1}}^*)^{b_{1\pi^{-1}}}, \dots, S^* (t_{k\pi^{-1}}^*)^{b_{k\pi^{-1}}}).$$

One checks that $\rho(\omega^x) = \rho(\omega)^{\varphi(x)}$ for all $\omega \in \Sigma$ and all $x \in G$, hence the actions of G and L on Ω and Λ^k , respectively, are permutation isomorphic. Finally, by identifying Λ^k with Δ^k , where Δ is the set of right cosets of S in T , we deduce that the permutation groups $G \leq \text{Sym}(\Omega)$ and $L \leq \text{Sym}(\Delta^k)$ are permutation isomorphic. \square

We also need the following easy lemma (the proof of [6, Theorem 4.1(e)] goes through unchanged).

Lemma 3.2. *Let $L \leq \text{Sym}(\Delta)$ be a finite $2'$ -elusive permutation group and let $K \leq S_k$ be a transitive subgroup, where $k \geq 2$. Then the product action of $L \text{ wr } K$ on Δ^k is also $2'$ -elusive.*

Lemma 3.3. *Let $N = T^k \leq \text{Sym}(\Omega)$ be a finite transitive permutation group with point stabiliser H , where T is simple and $k \geq 1$. Then N is $2'$ -elusive if and only if one of the following holds:*

- (i) $T = M_{11}$ and $H = \text{PSL}_2(11)^k$;
- (ii) $T = {}^2F_4(2)'$ and $H = \text{PSL}_2(25)^k$;
- (iii) $T = \text{PSL}_2(p)$ and $(C_p : C_r)^k \leq H < (C_p : C_{(p-1)/2})^k$, where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Proof. By applying Theorem 2.2(ii), we deduce that N is $2'$ -elusive if (i), (ii) or (iii) holds. For the remainder, let us assume N is $2'$ -elusive. First observe that H meets every N -class of elements of odd prime order. Since H is core-free in N , it follows that T is nonabelian (indeed, if T is abelian then $H = 1$ and $T = C_2$, which is incompatible with the fact that $|\Omega|$ is divisible by an odd prime).

Write $N = T_1 \times \dots \times T_k$ and $H_i = H \cap T_i$. Let $\pi_i : N \rightarrow T_i$ be the i -th projection map. If C is a conjugacy class of T then the corresponding subset of T_i is a conjugacy class of N , so H_i meets every T_i -class of elements of odd prime order. Since H is core-free in N , H_i is a proper subgroup of T_i and thus (T_i, H_i) is one of the cases arising in Theorem 2.2(ii). In particular, $T_i = M_{11}$, ${}^2F_4(2)'$ or $\text{PSL}_2(p)$ with p a Mersenne prime.

For each i we have $H_i \triangleleft \pi_i(H) < T_i$ (note that $\pi_i(H) < T_i$ since T_i is simple). If $T_i = M_{11}$ then $H_i = \text{PSL}_2(11)$ is a maximal subgroup of T_i , so in this case $H_i = \pi_i(H) = \text{PSL}_2(11)$ and thus $H = \text{PSL}_2(11)^k$ as in part (i) of the lemma. By the same argument, we deduce that $H = \text{PSL}_2(25)^k$ if $T_i = {}^2F_4(2)'$.

Finally, let us assume $T_i = \text{PSL}_2(p)$, where p is a Mersenne prime. Here

$$C_p : C_r \leq H_i \leq C_p : C_{(p-1)/2}$$

where r is the product of the distinct prime divisors of $(p-1)/2$. Since any overgroup of $C_p:C_r$ in $\text{PSL}_2(p)$ is contained in $C_p:C_{(p-1)/2}$, it follows that

$$C_p:C_r \leq H_i \leq \pi_i(H) \leq C_p:C_{(p-1)/2}.$$

Therefore H is as given in part (iii), and we note that $H < (C_p:C_{(p-1)/2})^k$ since $|\Omega|$ is divisible by an odd prime. \square

We are now in a position to prove Theorems 1 and 2.

Proof of Theorems 1 and 2. Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive quasiprimitive permutation group with socle N and point stabiliser $H = G_\alpha$. We claim that N is the unique minimal normal subgroup of G . To see this, suppose that N_1 and N_2 are distinct minimal normal subgroups of G . Then N_1 and N_2 commute, so they are regular and nonabelian by [10, Theorem 4.2A]. In particular, $N_1 \cong T^k$ for some nonabelian simple group T and positive integer k , so N contains derangements of odd prime order, but this is incompatible with the fact that G is $2'$ -elusive. Therefore, N is the unique minimal normal subgroup of G .

Write $N = T_1 \times \cdots \times T_k$ for some positive integer k such that $T_i \cong T$ for some simple group T . Since G is quasiprimitive, it follows that N is transitive and thus $2'$ -elusive, so Lemma 3.3 implies that one of the following holds (in particular, N is nonabelian):

- (a) $T = \text{M}_{11}$ and $N_\alpha = \text{PSL}_2(11)^k$;
- (b) $T = {}^2F_4(2)'$ and $N_\alpha = \text{PSL}_2(25)^k$;
- (c) $T = \text{PSL}_2(p)$ and $(C_p:C_r)^k \leq N_\alpha < (C_p:C_{(p-1)/2})^k$, where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Since N is the unique minimal normal subgroup of G , it follows that $C_G(N) = 1$ and thus $G \leq \text{Aut}(N) = \text{Aut}(T) \text{ wr } S_k$. Let $K \leq S_k$ be the group induced by G on the set of k simple direct factors of N . Then

$$T^k \leq G \leq \text{Aut}(T) \text{ wr } K$$

and the minimality of N implies that K is transitive. Also note that $G = NH$, so H also induces the group K on the set of k simple direct factors of N .

If (a) holds then $G = \text{M}_{11} \text{ wr } K$ is the only possibility (since $\text{Aut}(\text{M}_{11}) = \text{M}_{11}$), so $H = \text{PSL}_2(11) \text{ wr } K$. In view of Lemma 3.1, we may identify Ω with Δ^k , where Δ is the set of right cosets of $\text{PSL}_2(11)$ in M_{11} , so G is a primitive product-type group as in Theorem 1(i). In addition, we note that any group of this form is primitive and elusive (and therefore $2'$ -elusive since $|\Omega|$ is divisible by 3).

Next assume (b) holds. Set $X = N_{\text{Sym}(\Omega)}(N)$ and observe that $X = {}^2F_4(2) \text{ wr } S_k$ and $X_\alpha = (\text{PSL}_2(25).2_3) \text{ wr } S_k$. Since G induces the transitive subgroup $K \leq S_k$, it follows that $G \leq {}^2F_4(2) \text{ wr } K \leq X$ and $H = G \cap X_\alpha$. By applying Lemma 3.1 we can identify Ω with Δ^k , where Δ is the set of right cosets of $\text{PSL}_2(25)$ in ${}^2F_4(2)'$, and we see that G is a primitive product-type group as in Theorem 1(ii). By combining Theorem 2.3 and Lemma 3.2, we deduce that any primitive group of this form is indeed $2'$ -elusive.

Finally, suppose that (c) holds. Let $\pi_i : N \rightarrow T_i$ be the i -th projection map. For each $i \in \{1, \dots, k\}$, set $R_i = \pi_i(N_\alpha)$, so

$$C_p:C_r \leq R_i \leq C_p:C_{(p-1)/2} < T_i.$$

Since H normalises N_α and acts transitively on the set of k simple direct factors of N , it follows that $R_i \cong R_j$ for all i, j . Moreover, H normalises the subgroup $R = R_1 \times \cdots \times R_k$ of N . For each i , let $J_i = N_{T_i}(\text{O}_p(R_i)) = C_p:C_{(p-1)/2}$ and note that H normalises the subgroup $J = J_1 \times \cdots \times J_k$ of N . Moreover, $N_\alpha \leq R \leq J \leq N$ and N_α is a subdirect product of R . Also note that $N_\alpha \neq J$ since $|\Omega|$ is divisible by an odd prime. Therefore, $H < JH < G$ and thus G preserves a nontrivial system of imprimitivity \mathcal{P} of Ω such that the stabiliser of the block containing α is JH . Note that $JH \cap N = J$. The kernel of the action of G on \mathcal{P} is an intransitive normal subgroup of G , so this action is faithful by the quasiprimitivity of G .

Finally, by applying Lemma 3.1 we can identify \mathcal{P} with the Cartesian product Δ^k , where Δ is the set of right cosets of $C_p:C_{(p-1)/2}$ in $\text{PSL}_2(p)$.

This completes the proof of Theorems 1 and 2. \square

Remark 3.4. Let $G = \text{PSL}_2(p) \text{ wr } S_k$ and $H = (C_p:C_r) \text{ wr } S_k$, where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$. In addition, let us assume that $r \neq (p-1)/2$ (note that $p = 2^7 - 1$ is the smallest Mersenne prime with this property). Then the action of G on the set $\Omega = G/H$ of right cosets of H is quasiprimitive. Moreover, Lemma 3.1 implies that the action of G on Ω can be identified with the usual product action of G on Δ^k , where Δ is the set of right cosets of $C_p:C_r$ in $\text{PSL}_2(p)$. Then by applying Theorem 2.3 and Lemma 3.2, we deduce that the action of G on Ω is $2'$ -elusive. This shows that the set-up described in Theorem 2 does give rise to genuine examples.

4. BIQUASIPRIMITIVE GROUPS

In this section we turn our attention to biquasiprimitive permutation groups; our aim is to prove Theorem 3. Recall that a transitive permutation group $G \leq \text{Sym}(\Omega)$ is *biquasiprimitive* if every nontrivial normal subgroup of G has at most two orbits and there is some normal subgroup with two orbits Δ_1 and Δ_2 . Fix such a normal subgroup and let G^+ denote the index-two subgroup of G that fixes Δ_1 and Δ_2 setwise, so $\Omega = \Delta_1 \cup \Delta_2$ is a G -invariant partition of Ω .

Recall from the introduction that the elusive biquasiprimitive groups have been determined by Giudici and Xu (see [15, Theorem 1.4]). Our goal is to extend this result to $2'$ -elusive groups.

Lemma 4.1. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser H and let N be a minimal normal subgroup of G . Then $G^+ = NH$.*

Proof. If $N \leq G^+$ then the biquasiprimitivity of G implies that N acts transitively on each G^+ -orbit and thus $G^+ = NH$. Seeking a contradiction, suppose that $N \not\leq G^+$. Then by the minimality of N we have $N \cap G^+ = 1$. Since $|G : G^+| = 2$ it follows that $|N| = 2$ and $G = G^+ \times C_2$. Each orbit of N has size 2 and thus $|\Omega| \in \{2, 4\}$ since G is biquasiprimitive. But this contradicts the fact that $|\Omega|$ is divisible by an odd prime (because G is $2'$ -elusive). The result follows. \square

We now partition the proof of Theorem 3 into two parts, according to whether or not G^+ acts faithfully on its orbits Δ_1 and Δ_2 .

4.1. G^+ acts faithfully on both orbits.

Lemma 4.2. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser $H = G_\alpha$ and suppose that G^+ acts faithfully on its two orbits. Then G has a unique minimal normal subgroup $N = T^k$, where $k \geq 1$ and T, N_α are one of the following:*

- (i) $k = 1$, $T = A_6$ and $N_\alpha = A_5$;
- (ii) $T = M_{11}$ and $N_\alpha = \text{PSL}_2(11)^k$;
- (iii) $T = {}^2F_4(2)'$ and $N_\alpha = \text{PSL}_2(25)^k$;
- (iv) $T = \text{PSL}_2(p)$ and $(C_p:C_r)^k \leq N_\alpha < (C_p:C_{(p-1)/2})^k$ where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Proof. Let N be a minimal normal subgroup of G . By Lemma 4.1, $N \leq G^+$ and since G is biquasiprimitive, N acts transitively on both Δ_1 and Δ_2 . Moreover, since G is transitive and we are assuming that G^+ acts faithfully on Δ_1 and Δ_2 , it follows that $N^{\Delta_1} \cong N \cong N^{\Delta_2}$.

We claim that N is the unique minimal normal subgroup of G . To see this, suppose that M is another minimal normal subgroup of G , so N^{Δ_1} and M^{Δ_1} are both transitive normal subgroups of $(G^+)^{\Delta_1}$. Since $N^{\Delta_1} \cap M^{\Delta_1} = 1$ it follows that $[N^{\Delta_1}, M^{\Delta_1}] = 1$, so [10, Theorem 4.2A] implies that N^{Δ_1} and M^{Δ_1} are regular on Δ_1 and $N \cong M \cong T^k$ for some

finite nonabelian simple group T and positive integer k . Similarly, N and M act faithfully and regularly on Δ_2 and thus every element of odd prime order in N is a derangement on Ω . This is a contradiction since G is $2'$ -elusive, so N is the unique minimal normal subgroup of G as claimed. Write $N = T^k$, where T is simple and $k \geq 1$.

If N is abelian then it is semiregular on Ω with two orbits, so $T = C_2$ is the only possibility since G is $2'$ -elusive. But this implies that $|\Omega| = 2^{k+1}$, which is a contradiction since $|\Omega|$ is divisible by an odd prime. Therefore, T is nonabelian. Write $N = T_1 \times \cdots \times T_k$ with $T_i \cong T$, and let $\pi_i : N \rightarrow T_i$ be the i -th projection map. Note that $G \leq \text{Aut}(N) = \text{Aut}(T) \text{ wr } S_k$.

Fix $\alpha \in \Delta_1$ and set $H = G_\alpha$. Now each $n \in N$ of odd prime order fixes an element of Ω and is therefore G -conjugate to an element of N_α . Thus every $\text{Aut}(N)$ -class of elements of odd prime order in N meets N_α . Let $t \in T$ be an element of odd prime order. Then $(t, \dots, t) \in N$ is $\text{Aut}(N)$ -conjugate to an element of N_α , so for each $i \in \{1, \dots, k\}$ we see that $\pi_i(N_\alpha)$ meets every $\text{Aut}(T_i)$ -class of elements of odd prime order in T_i . Hence either $\pi_i(N_\alpha) = T_i$, or $(T_i, \pi_i(N_\alpha))$ is given by Theorem 2.2(i). Similarly, $(t, 1, \dots, 1) \in N$ is also $\text{Aut}(N)$ -conjugate to an element of N_α , so

$$1 \neq N_\alpha \cap T_\ell \leq \pi_\ell(N_\alpha) \leq T_\ell \quad (1)$$

for some $\ell \in \{1, \dots, k\}$. Since N is faithful on Δ_1 it follows that $T_\ell \not\leq N_\alpha$ and so the fact that T_ℓ is simple implies that $\pi_\ell(N_\alpha) \neq T_\ell$. In particular, $(T_\ell, \pi_\ell(N_\alpha))$ is one of the cases in Theorem 2.2(i). Set $R = \pi_\ell(N_\alpha)$.

The transitivity of N on Δ_1 implies that $G^+ = NH$, so G^+ and H have the same orbits on $\{T_1, \dots, T_k\}$. The minimality of N implies that G acts transitively on this set, so G^+ is either transitive or has two equal sized orbits (since $|G : G^+| = 2$). Let \mathcal{O}_1 be the orbit of G^+ on $\{T_1, \dots, T_k\}$ containing T_ℓ (where ℓ is the integer in (1)). Without loss of generality we may assume that $\{T_1, \dots, T_{\lfloor k/2 \rfloor}\} \subseteq \mathcal{O}_1$. Note that $\pi_i(N_\alpha) \cong \pi_\ell(N_\alpha)$ and $N_\alpha \cap T_i \cong N_\alpha \cap T_\ell$ for all $i \in \mathcal{O}_1$. We now consider two cases.

Case 1. $T \neq \text{PSL}_2(p)$

Suppose first that $T \neq \text{PSL}_2(p)$, in which case $\pi_\ell(N_\alpha)$ is simple (see Table 1). In view of (1), it follows that $N_\alpha \cap T_\ell = \pi_\ell(N_\alpha) = R$. We claim that $N_\alpha \cong R^k$. This is clear if $\mathcal{O}_1 = \{T_1, \dots, T_k\}$, so let us assume that k is even and $\mathcal{O}_1 = \{T_1, \dots, T_{k/2}\}$, so $R_1 \times \cdots \times R_{k/2} \leq N_\alpha$, where $R_i = \pi_i(N_\alpha) \cong R$ and $R_i < T_i$ for each $i \in \{1, \dots, k/2\}$.

Let $t \in T$ be an element of odd prime order and set $g = (t, 1, \dots, 1, t) \in N$. Since

$$\{T_1, \dots, T_k\} = \{T_1, \dots, T_{k/2}\} \cup \{T_{k/2+1}, \dots, T_k\}$$

is a G -invariant partition, it follows that every G -conjugate of g has precisely one nontrivial entry in the first $k/2$ coordinates and precisely one nontrivial entry in the last $k/2$ coordinates. Since G is $2'$ -elusive, g is conjugate to an element of N_α . By multiplying this conjugate by an appropriate element of $R_1 \times \cdots \times R_{k/2}$, we deduce that N_α contains an element with precisely one nontrivial entry, which occurs in the last $k/2$ coordinates. Hence $1 \neq N_\alpha \cap T_i \leq \pi_i(N_\alpha)$ for all $i \in \{k/2 + 1, \dots, k\}$. By arguing as above we deduce that $\pi_i(N_\alpha) \cong R$ and thus $N_\alpha \cong R^k$. This justifies the claim.

We now consider the possibilities for T arising in Theorem 2.2(i). If $T = \Omega_8^+(2)$ or $\text{P}\Omega_8^+(3)$ then the proof of [15, Proposition 4.6] produces a derangement of order 5 in N . Similarly, if $T = A_6$ and $k \geq 2$ then the same proof gives a derangement of order 3 (if $k = 1$ then case (i) holds). If $T = \text{M}_{11}$ or ${}^2F_4(2)'$ then we are in case (ii) or (iii), respectively.

Case 2. $T = \text{PSL}_2(p)$, where p is a Mersenne prime

Finally, let us assume that $T = \text{PSL}_2(p)$ with p a Mersenne prime. We have seen that for each element $t \in T$ of odd prime order, $(t, 1, \dots, 1)$ is G -conjugate to an element of N_α , and that the unique nontrivial entry of this element lies in $\mathcal{O}_1 \subseteq \{T_1, \dots, T_k\}$. Since H acts transitively on the set $\{N_\alpha \cap T_i \mid T_i \in \mathcal{O}_1\}$, it follows that $N_\alpha \cap T_i$ meets each $\text{Aut}(T_i)$ -class

of elements of odd prime order in T_i , for all $i \in \mathcal{O}_1$. This immediately implies that (iv) holds if $\mathcal{O}_1 = \{T_1, \dots, T_k\}$.

To complete the proof, we may assume k is even and $\mathcal{O}_1 = \{T_1, \dots, T_{k/2}\}$. The above argument shows that $Q_1 \times \dots \times Q_{k/2} \leq N_\alpha$, where $Q_i \cong C_p : C_r$ for all i (here r is the product of the distinct prime divisors of $(p-1)/2$). Moreover, any $n \in N_\alpha$ of odd prime order projects onto an element of Q_i for each $i \in \{1, \dots, k/2\}$.

Let $t \in T$ be an element of odd prime order and set $g = (t, 1, \dots, 1, t) \in N$. As observed above, each G -conjugate of g has precisely one nontrivial entry in the first $k/2$ coordinates and one in the last $k/2$ coordinates. An appropriate G -conjugate of g is contained in N_α , which we can multiply by an element of $Q_1 \times \dots \times Q_{k/2} \leq N_\alpha$ to obtain an element of odd prime order in N_α with precisely one nontrivial entry in the i -th coordinate for some $i \in \{k/2 + 1, \dots, k\}$. Therefore $1 \neq N_\alpha \cap T_i \leq \pi_i(N_\alpha)$. Since T_i is simple and $\pi_i(N_\alpha)$ meets every $\text{Aut}(T_i)$ -class of elements of odd prime order in T_i , it follows that $\pi_i(N_\alpha) \leq C_p : C_{(p-1)/2}$. Moreover, we also see that $N_\alpha \cap T_i$ meets every $\text{Aut}(T_i)$ -class of elements of odd prime order in T_i , so $C_p : C_r \leq N_\alpha \cap T_i$. Therefore,

$$C_p : C_r \leq N_\alpha \cap T_i \leq \pi_i(N_\alpha) \leq C_p : C_{(p-1)/2}$$

for all $i \in \{k/2 + 1, \dots, k\}$, and we conclude that (iv) holds. \square

Theorem 4.3. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser $H = G_\alpha$ and socle N . Let $K \leq S_k$ be the transitive group induced by G on the set of k simple direct factors of N and let $K^+ \leq K$ be the group induced by G^+ . Assume that G^+ acts faithfully on its two orbits. Then one of the following holds:*

- (i) $(G, H) = (M_{10}, A_5)$ or $(\text{Aut}(A_6), S_5)$;
- (ii) $G = M_{11} \text{ wr } K$, $H = \text{PSL}_2(11) \text{ wr } K^+$ and $|K : K^+| = 2$;
- (iii) $N = ({}^2F_4(2)')^k \leq G \leq {}^2F_4(2) \text{ wr } K$ and $H = N_{G^+}(N_\alpha)$, where $N_\alpha = \text{PSL}_2(25)^k$;
- (iv) $N = \text{PSL}_2(p)^k \leq G \leq \text{PGL}_2(p) \text{ wr } K$,

$$(C_p : C_r)^k \leq N_\alpha < (C_p : C_{(p-1)/2})^k$$

and

$$H < N_{G^+}(N_\alpha) = G^+ \cap ((C_p : C_{p-1}) \text{ wr } K),$$

where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Moreover, each group G in (i), (ii) and (iii) is $2'$ -elusive and biquasiprimitive.

Proof. By Lemma 4.2, N is the unique minimal normal subgroup of G and we may write $N = T^k$ with $k \geq 1$, where the possibilities for T and N_α are described in the lemma. Note that $G \leq \text{Aut}(N) = \text{Aut}(T) \text{ wr } S_k$.

First assume that $T = A_6$ and $N_\alpha = A_5$, so $G \leq \text{Aut}(A_6)$. Since G is $2'$ -elusive, each element in N of odd prime order is G -conjugate to an element of N_α , so G must contain an outer automorphism of S_6 . Therefore $G = M_{10}$ or $\text{Aut}(A_6)$, and N has two orbits of size 6. It follows that $(G, H) = (M_{10}, A_5)$ or $(\text{Aut}(A_6), S_5)$, as in (i). It is easy to check that G is indeed $2'$ -elusive and biquasiprimitive in both cases.

Next assume that $T = M_{11}$ and $N_\alpha = \text{PSL}_2(11)^k$. Since $\text{Aut}(T) = T$ it follows that $G = T \text{ wr } K$ for some transitive subgroup $K \leq S_k$ (the transitivity of K follows from the minimality of N). Moreover, $G^+ = T \text{ wr } K^+$ and $H = \text{PSL}_2(11) \text{ wr } K^+$, where $|K : K^+| = 2$. This is case (ii) in the statement of the theorem.

We claim that every group G as in (ii) is $2'$ -elusive and biquasiprimitive. First observe that $N = T^k$ is the unique minimal normal subgroup of G and N has two orbits on Ω . Therefore, if M is any nontrivial normal subgroup of G then $N \leq M$, so M has at most two orbits on Ω and thus G is biquasiprimitive. To see that G is $2'$ -elusive, let Δ denote the set of right cosets of $\text{PSL}_2(11)$ in T . Since $N_\alpha = \text{PSL}_2(11)^k$ and $G^+ = T \text{ wr } K^+$ acts transitively and faithfully on Δ_1 , we may identify Δ_1 with the Cartesian product Δ^k so that G^+ acts

on Δ^k with its standard product action (see Lemma 3.1). The action of T on Δ is elusive, so [6, Theorem 4.1(e)] implies that the action of G^+ on Δ_1 is also elusive. Therefore, each $g \in G^+$ of prime order has fixed points on Δ_1 , and hence on Ω . Since every element in G of odd prime order lies in G^+ , we deduce that G is indeed $2'$ -elusive.

Next suppose that $T = {}^2F_4(2)'$ and $N_\alpha = \text{PSL}_2(25)^k$. The minimality of N implies that G induces a transitive group $K \leq S_k$ on the set of k simple direct factors of N , so $N \trianglelefteq G \leq {}^2F_4(2) \text{ wr } K$ (note that $\text{Aut}({}^2F_4(2)') = {}^2F_4(2) = {}^2F_4(2)'.2$) and G/N is a subgroup of $C_2 \text{ wr } K$ that projects onto K . Note that $H \leq G^+$ and G^+/N is an index-two subgroup of G/N . Since G^+ acts transitively and faithfully on Δ_1 , by Lemma 3.1 we may identify Δ_1 with Δ^k , where Δ is the set of right cosets of $\text{PSL}_2(25)$ in T , so that G^+ acts on Δ^k via the usual product action. In particular, $H = N_{G^+}(N_\alpha)$ as in part (iii). By arguing as above, we see that every group G as in (iii) is biquasiprimitive. Also note that the action of T on Δ is $2'$ -elusive (see Theorem 1), so Lemma 3.2 implies that the action of G^+ on Δ_1 is also $2'$ -elusive and we conclude that G is $2'$ -elusive as above.

Finally, let us assume that $T = \text{PSL}_2(p)$ and

$$(C_p : C_r)^k \leq N_\alpha < (C_p : C_{(p-1)/2})^k$$

where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$. Here $N \trianglelefteq G \leq \text{PGL}_2(p) \text{ wr } K$ for some transitive subgroup $K \leq S_k$, as in case (iv), and G/N is a subgroup of $C_2 \text{ wr } K$ that projects onto K . In addition, we note that $H < N_{G^+}(N_\alpha)$ since $N_\alpha \trianglelefteq H$ and $|G^+ : N_{G^+}(N_\alpha)|$ is a power of 2. \square

Remark 4.4. Let G be a group as in case (iii) of Theorem 4.3. In general, there is more than one possibility for $H = G_\alpha$ with the desired property that the action of G on G/H is $2'$ -elusive and biquasiprimitive. For example, if $G = \langle ({}^2F_4(2)')^k, (g, \dots, g) \rangle : S_k$, where ${}^2F_4(2) = \langle {}^2F_4(2)', g \rangle$ and $k \geq 2$, then we can take $H = N_{G^+}(N_\alpha)$ with

$$G^+ = \langle ({}^2F_4(2)')^k, (g, \dots, g) \rangle : A_k \text{ or } {}^2F_4(2)' \text{ wr } S_k$$

and $N_\alpha = \text{PSL}_2(25)^k$.

Remark 4.5. Examples do occur in case (iv) of Theorem 4.3. To see this, fix a Mersenne prime p such that $r < (p-1)/2$. Then the almost simple group $G = \text{PGL}_2(p)$ with $H = C_p : C_r$ as in Theorem 2.3 is biquasiprimitive and $2'$ -elusive (note that $G^+ = \text{PSL}_2(p)$). Similarly, if $k \geq 2$ then we can take $G = \text{PSL}_2(p) \text{ wr } S_k$ and $H = (C_p : C_r) \text{ wr } A_k$ (here $G^+ = \text{PSL}_2(p) \text{ wr } A_k$).

4.2. G^+ is not faithful on both orbits. To complete the proof of Theorem 3, we may assume that G^+ is not faithful on at least one of its two orbits Δ_1 and Δ_2 on Ω . We begin with a lemma that describes the structure of $\text{soc}(G)$ and $\text{soc}(G)_\alpha$.

Lemma 4.6. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser $H = G_\alpha$ and assume that G^+ is not faithful on at least one of its orbits. Then G has a unique minimal normal subgroup $N = T^k$, where $k \geq 2$ is even and T, N_α are one of the following:*

- (i) $T = M_{11}$ and $N_\alpha = \text{PSL}_2(11)^{k/2} \times M_{11}^{k/2}$;
- (ii) $T = {}^2F_4(2)'$ and $N_\alpha = \text{PSL}_2(25)^{k/2} \times ({}^2F_4(2)')^{k/2}$;
- (iii) $T = \text{PSL}_2(p)$ and

$$(C_p : C_r)^{k/2} \times \text{PSL}_2(p)^{k/2} \leq N_\alpha < (C_p : C_{(p-1)/2})^{k/2} \times \text{PSL}_2(p)^{k/2},$$

where p is a Mersenne prime and r is the product of the distinct prime divisors of $(p-1)/2$.

Proof. We adapt the proof of [15, Lemma 4.7]. Without loss of generality, we may assume that the action of G^+ on Δ_1 is not faithful. Let M_1 be a minimal normal subgroup of G^+ contained in the kernel of the action of G^+ on Δ_1 . Fix an element $g \in G \setminus G^+$ and

observe that M_1^g is a minimal normal subgroup of G^+ contained in the kernel of the action of G^+ on Δ_2 (in particular, G^+ is not faithful on either orbit). Since G is faithful on Ω , it follows that $M_1 \cap M_1^g = 1$, M_1 is faithful on Δ_2 and M_1^g is faithful on Δ_1 . In addition, M_1 acts transitively on Δ_2 , and M_1^g acts transitively on Δ_1 (since G is biquasiprimitive). Since $g^2 \in G^+$ we deduce that $(M_1^g)^g = M_1$ and so $N = M_1 \times M_1^g$ is a minimal normal subgroup of G . Moreover, if $h \in M_1$ is a derangement on Δ_2 , then $(h, h^g) \in N$ is a derangement on Ω . Therefore M_1 is 2'-elusive on Δ_2 , so the possibilities for M_1 and $(M_1)_\alpha$ are given by Lemma 3.3 (where $\alpha \in \Delta_2$). It follows that $N = T^k$ for some even integer $k \geq 2$, and

$$N_\alpha = (M_1)_\alpha \times (M_1)^g \cong (M_1)_\alpha \times T^{k/2}.$$

Therefore, to complete the proof of the lemma it remains to show that N is the unique minimal normal subgroup of G . Set $L = (G^+)^{\Delta_2}$ and note that

$$\text{soc}(G) \leq \text{soc}(L) \times \text{soc}(L)$$

by [23, Lemma 3.2(a)].

First assume that $T = M_{11}$ or ${}^2F_4(2)'$. Then $(M_1)_\alpha$ is self-normalising in M_1 , so [10, Theorem 4.2A] implies that $C_{\text{Sym}(\Delta_2)}(M_1) = 1$ and thus $\text{soc}(L) = M_1$. We conclude that N is the unique minimal normal subgroup of G .

Finally, let us assume that $T = \text{PSL}_2(p)$ with p a Mersenne prime. As usual, let r be the product of the distinct prime divisors of $(p-1)/2$. Then

$$(C_p : C_r)^{k/2} \leq (M_1)_\alpha \leq (C_p : C_{(p-1)/2})^{k/2}$$

and we note that $C_p : C_r \trianglelefteq C_p : C_{(p-1)/2}$ and $C_p : C_{(p-1)/2}$ is a maximal subgroup of T . In particular, $N_{M_1}((M_1)_\alpha) = (C_p : C_{(p-1)/2})^{k/2}$ has odd order. Therefore, [10, Theorem 4.2A] implies that $C_L(M_1)$ has odd order and is semiregular on Δ_2 .

If $C_L(M_1) = 1$ then $\text{soc}(L) = M_1$ and so N is the unique minimal normal subgroup of G . Now assume $C_L(M_1) \neq 1$. Let J be a minimal normal subgroup of L that is contained in $C_L(M_1)$. Since $C_L(M_1)$ has odd order, J is elementary abelian. However, $|\Delta_2|$ is divisible by $p+1$, which is a power of 2, and so J is intransitive on Δ_2 . In particular, L is not quasiprimitive on Δ_2 . Moreover, since G^+ is not faithful on its orbits, [23, Lemma 3.5] implies that the structure of G is as in case (b) of [23, Theorem 1.1]. In particular, M_1 is the unique transitive minimal normal subgroup of L and $\text{soc}(G) = M_1 \times (M_1)^g = N$, so N is the unique minimal normal subgroup of G . \square

We are now in a position to complete the proof of Theorem 3. In the statement and proof of Theorem 4.7, we will write R for the almost simple maximal subgroup $\text{PSL}_2(25).2_3 < {}^2F_4(2)$ (this is a nonsplit extension).

Theorem 4.7. *Let $G \leq \text{Sym}(\Omega)$ be a finite 2'-elusive biquasiprimitive permutation group with point stabiliser $H = G_\alpha$ and socle N . Let $K \leq S_k$ be the transitive group induced by G on the set of k simple direct factors of N and let $K^+ \leq K$ be the group induced by G^+ . Assume that G^+ is not faithful on at least one of its orbits. Then k is even, K^+ is intransitive, $|K : K^+| = 2$ and one of the following holds:*

- (i) $G = M_{11} \text{ wr } K$ and $H = (\text{PSL}_2(11)^{k/2} \times M_{11}^{k/2}) : K^+$;
- (ii) $N = ({}^2F_4(2)')^k \trianglelefteq G \leq {}^2F_4(2) \text{ wr } K$, $N_\alpha = \text{PSL}_2(25)^{k/2} \times ({}^2F_4(2)')^{k/2}$ and

$$H = N_{G^+}(N_\alpha) = G^+ \cap (R^{k/2} \times {}^2F_4(2)^{k/2}) : K^+$$

with $G^+ = G \cap ({}^2F_4(2) \text{ wr } K^+)$;

- (iii) $N = \text{PSL}_2(p)^k \trianglelefteq G \leq \text{PGL}_2(p) \text{ wr } K$,

$$(C_p : C_r)^{k/2} \times \text{PSL}_2(p)^{k/2} \leq N_\alpha < (C_p : C_{(p-1)/2})^{k/2} \times \text{PSL}_2(p)^{k/2}$$

and

$$H < N_{G^+}(N_\alpha) = G^+ \cap ((C_p : C_{p-1})^{k/2} \times \text{PGL}_2(p)^{k/2}) : K^+,$$

where $G^+ = G \cap (\text{PGL}_2(p) \text{ wr } K^+)$ and p is a Mersenne prime.

Moreover, each group G in (i) and (ii) is 2'-elusive and biquasiprimitive.

Proof. By Lemma 4.6, G has a unique minimal normal subgroup $N = T^k$, where $k \geq 2$ is even and the possibilities for T and N_α are described in the lemma. In particular, we note that $G \leq \text{Aut}(N) = \text{Aut}(T) \text{ wr } S_k$. Write $N = T_1 \times \cdots \times T_k$ with $T_i \cong T$ for each i , and let $K \leq S_k$ be the permutation group induced by the conjugation action of G on $\{T_1, \dots, T_k\}$. Note that K is transitive since N is minimal. Moreover, $G \leq \text{Aut}(T) \text{ wr } K$.

Now $G^+ = NH$ and $H \leq N_{G^+}(N_\alpha)$, so G^+ has two orbits on $\{T_1, \dots, T_k\}$ and it induces an intransitive index-two subgroup $K^+ < K$. Note that $G^+ = G \cap (\text{Aut}(T) \text{ wr } K^+)$. We may assume that the orbits of K^+ are $\{T_1, \dots, T_{k/2}\}$ and $\{T_{k/2+1}, \dots, T_k\}$. We now consider the three cases arising in Lemma 4.6.

First assume that $T = M_{11}$ and $N_\alpha = \text{PSL}_2(11)^{k/2} \times M_{11}^{k/2}$. Since $\text{Aut}(M_{11}) = M_{11}$ it follows that $G = M_{11} \text{ wr } K$, so $G^+ = M_{11} \text{ wr } K^+$ and since $G^+ = NH$ we deduce that $H = (\text{PSL}_2(11)^{k/2} \times M_{11}^{k/2}) : K^+$ as in case (i). Now each $g \in G^+$ of prime order is conjugate to an element of $\text{PSL}_2(11) \text{ wr } K^+$ (this follows from [6, Theorem 4.1(e)]), which is contained in H , so any group G of this form is 2'-elusive since every element in G of odd prime order is contained in G^+ . In addition, G is biquasiprimitive since every nontrivial normal subgroup of G contains N , which is transitive on Δ_1 and Δ_2 .

Next assume $T = {}^2F_4(2)'$ and $N_\alpha = \text{PSL}_2(25)^{k/2} \times ({}^2F_4(2)')^{k/2}$. Here $G \leq {}^2F_4(2) \text{ wr } K$, $G^+ = G \cap ({}^2F_4(2) \text{ wr } K^+)$ and

$$H \leq N_{G^+}(N_\alpha) = G^+ \cap (R^{k/2} \times {}^2F_4(2)^{k/2}) : K^+$$

as in (ii). Since $\text{PSL}_2(25)$ is a maximal subgroup of ${}^2F_4(2)'$, and G^+ acts transitively on $\{T_1, \dots, T_{k/2}\}$, it follows that G^+ acts primitively on Δ_1 and Δ_2 , inducing a subgroup of ${}^2F_4(2) \text{ wr } S_{k/2}$ on each orbit. Therefore H is a maximal subgroup of G^+ and thus $H = N_{G^+}(N_\alpha)$. As in the previous case, any such group G is biquasiprimitive. Moreover, every element in G of odd prime order is contained in G^+ , and Theorem 1 implies that every element of odd prime order in G^+ is conjugate to an element of $\text{PSL}_2(25) \text{ wr } K^+$, which is contained in H . We conclude that G is 2'-elusive.

Finally, let us assume that $T = \text{PSL}_2(p)$ with p a Mersenne prime. Here we have $G \leq \text{PGL}_2(p) \text{ wr } K$, $G^+ = G \cap (\text{PGL}_2(p) \text{ wr } K^+)$ and

$$H < N_{G^+}(N_\alpha) = G^+ \cap (C_p : C_{p-1})^{k/2} \times \text{PGL}_2(p)^{k/2} : K^+$$

as in (iii). Note that H is a proper subgroup of $N_{G^+}(N_\alpha)$ since $|G : N_{G^+}(N_\alpha)|$ is a power of 2 and $|\Omega| = |G : H|$ is divisible by an odd prime. \square

This completes the proof of Theorem 3.

Remark 4.8. Examples of 2'-elusive groups in case (iii) do exist. For instance, if p is a Mersenne prime with $r < (p-1)/2$, then we can take

$$G = \text{PSL}_2(p) \text{ wr } (S_{k/2} \text{ wr } S_2), \quad H = ((C_p : C_r)^{k/2} \times \text{PSL}_2(p)^{k/2}) : S_{k/2}^2,$$

where $S_{k/2} \text{ wr } S_2$ acts imprimitively on k points.

5. ARC-TRANSITIVE GRAPHS OF PRIME VALENCY

In this final section we will use Theorems 1, 2 and 3 to determine the 2'-elusive quasiprimitive and biquasiprimitive groups with a prime subdegree. As explained in the Introduction, this is the main step in the proof of Theorem 5, which we anticipate will play a key role in the proof of the Polycirculant Conjecture for arc-transitive graphs of valency $2p$, where p is a prime. We start by recalling some standard terminology.

Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group and let $\alpha \in \Omega$. Recall that the orbits of G_α on $\Omega \setminus \{\alpha\}$ are called *suborbits*, and the lengths of these orbits are the *subdegrees*

of G . It is well known that there is a one-to-one correspondence between the set of suborbits of G and the set of digraphs with vertex set Ω on which G acts arc-transitively. More precisely, the suborbit corresponding to a given digraph Γ is the set $\Gamma^+(\alpha)$ of out-neighbours of α in Γ . On the other hand, if β^{G_α} is the suborbit containing β then the corresponding digraph on Ω has arc-set

$$(\alpha, \beta)^G = \{(\alpha^g, \beta^g) \mid g \in G\},$$

which is simply the orbit of (α, β) with respect to the natural action of G on $\Omega \times \Omega$. Such an arc-set is called an *orbital* of G , and the corresponding digraph is referred to as an *orbital digraph*. Further, we say that the suborbit β^{G_α} , and also the orbital $(\alpha, \beta)^G$, is *self-paired* if there is an element $g \in G$ that interchanges α and β . In this situation, $(\omega_1, \omega_2) \in (\alpha, \beta)^G$ if and only if $(\omega_2, \omega_1) \in (\alpha, \beta)^G$, in which case the corresponding digraph Γ is a graph since we can ignore the directions on the edges (note that Γ is $|\beta^{G_\alpha}|$ -regular).

Note that if $g \in G$ interchanges α and β then it lies in $N_G(G_\alpha \cap G_\beta)$, but not in G_α . Also note that if $\alpha^g = \beta$ then $G_{\alpha, \beta} = G_\alpha \cap (G_\alpha)^g$ and $|\beta^{G_\alpha}| = |G_\alpha : G_\alpha \cap (G_\alpha)^g|$. We will need the following two lemmas (the proofs are easy exercises).

Lemma 5.1. *Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group and let β^{G_α} be a self-paired suborbit of G . Then the corresponding orbital graph is connected if and only if $G = \langle G_\alpha, g \rangle$ for each $g \in G$ that interchanges α and β .*

Lemma 5.2. *Let $G \leq \text{Sym}(\Omega)$ be a transitive imprimitive permutation group with system of imprimitivity \mathcal{P} . Let $\alpha, \omega \in \Omega$ and let $A \in \mathcal{P}$ be the block containing ω . Then the following hold:*

- (i) $|A^{G_\alpha}|$ divides $|\omega^{G_\alpha}|$;
- (ii) If $\alpha \in A$ then the orbital digraph corresponding to ω^{G_α} is disconnected.

In order to prove Theorem 5, we need to determine the $2'$ -elusive quasiprimitive and biquasiprimitive groups with a self-paired suborbit of prime length which satisfies the connectedness condition in Lemma 5.1. To get started, in the next lemma we determine the subdegrees of the relevant almost simple groups (see Theorem 2.3). Note that in the statement of the lemma, we write ℓ^k to denote that ℓ occurs as a subdegree k times.

Lemma 5.3. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive quasiprimitive or biquasiprimitive almost simple group with point stabiliser H . Let \mathcal{S} be the multiset of subdegrees of G . Then G has a prime subdegree if and only if one of the following holds:*

- (i) $(G, H) = (M_{11}, \text{PSL}_2(11))$, $\mathcal{S} = \{1, 11\}$;
- (ii) $(G, H) = (M_{10}, A_5)$ or $(\text{Aut}(A_6), S_5)$, $\mathcal{S} = \{1, 5, 6\}$;
- (iii) $(G, H) = (\text{PSL}_2(p), C_p : C_s)$ or $(\text{PGL}_2(p), C_p : C_{2s})$, $\mathcal{S} = \{1^{(p-1)/2s}, p^{(p-1)/2s}\}$, p is a Mersenne prime, $C_r \leq C_s < C_{(p-1)/2}$ and r is the product of the distinct prime divisors of $(p-1)/2$;
- (iv) $(G, H) = (\text{PGL}_2(p), C_p : C_s)$, $\mathcal{S} = \{1^{(p-1)/s}, p^{(p-1)/s}\}$, where p, s are as in part (iii).

Moreover, if Δ is a suborbit of prime length and Γ is the corresponding orbital digraph, then Δ is self-paired and Γ is connected, with the exception of case (ii).

Proof. The first two cases can be easily checked using MAGMA [3]. Note that in case (i), Γ is the complete graph K_{12} . Similarly, one checks that

$$\mathcal{S} = \begin{cases} \{1, 78, 300^2, 325^2, 975\} & (G, H) = ({}^2F_4(2)', \text{PSL}_2(25)) \text{ or } ({}^2F_4(2), \text{PSL}_2(25).2_3) \\ \{1^2, 78^2, 300^4, 325^4, 975^2\} & (G, H) = ({}^2F_4(2), \text{PSL}_2(25)) \end{cases}$$

so no cases with $\text{soc}(G) = {}^2F_4(2)'$ arise.

In view of Theorem 2.3, we may assume that $\text{soc}(G) = \text{PSL}_2(p)$ with p a Mersenne prime. First consider case (iii), with $(G, H) = (\text{PSL}_2(p), C_p : C_s)$. Let $N = N_G(H) = C_p : C_{(p-1)/2}$. By [10, Theorem 4.2A(i)], H has $|N_G(H) : H| = (p-1)/2s$ fixed points on Ω . Note that

the action of G on G/N is 2-transitive with degree $p + 1$, so in this action G has a unique suborbit of length p and thus $N \cap N^g = C_{(p-1)/2}$ for all $g \in G \setminus N$. Fix an element $g \in G \setminus N$ and let L be the unique subgroup of $N \cap N^g$ of order s , so $H \cap H^g \leq L$. Now N contains p cyclic subgroups of order s , each of which is contained in H , so $L \leq H$ and $L \leq H^g$. We conclude that $H \cap H^g \cong C_s$ for all $g \in G \setminus N$, so $\mathcal{S} = \{1^{(p-1)/2s}, p^{(p-1)/2s}\}$ as claimed.

Let $\Delta = \beta^{G_\alpha}$ be a suborbit of length p , so $G_\alpha \cap G_\beta \cong C_s$. Note that $G_\alpha \cap G_\beta$ fixes the $(p-1)/2s$ fixed points of G_α , and it also fixes one point from each of the $(p-1)/2s$ orbits of G_α of length p . Set $M = N_G(G_\alpha \cap G_\beta) = D_{p-1}$ and note that each element of M maps α and β to points fixed by $G_\alpha \cap G_\beta$. Also note that $M \cap N = C_{(p-1)/2}$ transitively permutes the fixed points of G_α and the set of orbits of G_α of length p . Thus M has at most two orbits on the set of $(p-1)/s$ fixed points of $G_\alpha \cap G_\beta$. In fact, N is the stabiliser in G of the set of fixed points of G_α (since N is maximal in G), and thus M is transitive on the set of fixed points of $G_\alpha \cap G_\beta$. In particular, each involution in $M \setminus (M \cap N)$ interchanges α with a fixed point of M contained in a suborbit of length p . Therefore, there exists an element $g \in M \setminus (M \cap N)$ that interchanges α and β , so Δ is self-paired. Moreover, since $g \notin N$ and N is the unique maximal subgroup of G containing G_α , it follows that $G = \langle G_\alpha, g \rangle$ and thus the corresponding orbital graph is connected by Lemma 5.1.

A similar argument applies when $G = \text{PGL}_2(p)$ in (iii) or (iv). We omit the details. \square

Lemma 5.4. *Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group such that*

$$N = \text{soc}(L)^k \trianglelefteq G \leq L \text{ wr } K,$$

$k \geq 2$ and G acts with its product action on $\Omega = \Delta^k$. Here $L \leq \text{Sym}(\Delta)$ is transitive and almost simple, $K \leq S_k$ is the group induced by G on the set of k simple direct factors of N . Assume that the following conditions are satisfied:

- (a) *N is transitive on Ω ;*
- (b) *The only element of Δ fixed by $\text{soc}(L)_\delta$ is δ ;*
- (c) *Either K is transitive, or it has two equal sized orbits on the set of k simple direct factors of N .*

Then G has a prime subdegree only if $k = 2$, $K = 1$ and $\text{soc}(L)$ has a prime subdegree on Δ . Moreover, if G has a self-paired suborbit of prime length, then the corresponding orbital graph is disconnected.

Proof. Let $\alpha = (\delta, \dots, \delta) \in \Omega$ and suppose ω^{G_α} is a self-paired suborbit of prime length, where $\omega = (\omega_1, \dots, \omega_k) \in \Omega \setminus \{\alpha\}$. Note that ω differs from α in at least one coordinate and since the only element of Δ fixed by $\text{soc}(L)_\delta$ is δ , it follows that α is the only element of Ω fixed by $N_\alpha = (\text{soc}(L)_\delta)^k$. Therefore $|\omega^{N_\alpha}| > 1$ divides $|\omega^{G_\alpha}|$ and thus $\omega^{N_\alpha} = \omega^{G_\alpha}$.

For each element $\beta = (\beta_1, \dots, \beta_k) \in \Omega$, we define the *support* of β to be the set $\{i \mid \beta_i \neq \delta\}$. Let I be the support of ω , and note that I is also the support of each element of ω^{N_α} . Now $|\omega^{N_\alpha}| = \prod_{i \in I} |\omega_i^{\text{soc}(L)_\delta}|$ and each term in the product is greater than 1 since $\omega_i \neq \delta$. But $|\omega^{N_\alpha}| = |\omega^{G_\alpha}|$ is a prime, so we must have $I = \{i\}$ for some i .

Since N is transitive on Ω we have $G = NG_\alpha$ and so G_α also induces the group K on the set of k simple direct factors of N . Therefore, for each j in the K -orbit i^K , there is an element of ω^{G_α} whose support is $\{j\}$. Since $\omega^{N_\alpha} = \omega^{G_\alpha}$ it follows that $i^K = i$ and thus $k = 2$ and $K = 1$, so $\text{soc}(L) \times \text{soc}(L) \trianglelefteq G \leq L \times L$.

Without loss of generality, we may assume that $\omega = (\gamma, \delta)$ where $|\gamma^{\text{soc}(L)_\delta}|$ is a prime. If $g = (g_1, g_2) \in G$ interchanges α and ω , then $\delta^{g_2} = \delta$. Therefore, $\delta^{h_2} = \delta$ for each $(h_1, h_2) \in \langle G_\alpha, g \rangle$ and thus $G \neq \langle G_\alpha, g \rangle$. In particular, Lemma 5.1 implies that the orbital graph corresponding to ω^{G_α} is disconnected. \square

Lemma 5.5. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive quasiprimitive permutation group with a non-simple socle. Then G does not have a prime subdegree that corresponds to a connected orbital graph.*

Proof. Write $N = \text{soc}(G) = \text{soc}(L)^k$, where $k \geq 2$ and $L \leq \text{Sym}(\Delta)$ is the transitive almost simple group described in Theorems 1 and 2. Let $\alpha \in \Omega$. Seeking a contradiction, suppose that ω^{G_α} is a self-paired suborbit of prime length and the corresponding orbital graph is connected.

If G is primitive then Theorem 1 implies that G induces a transitive permutation group on the set of simple direct factors of its socle, so G does not have a prime subdegree by Lemma 5.4. For the remainder, we may assume that G is imprimitive, in which case the structure of G is described in Theorem 2. In particular, $\text{soc}(L) = \text{PSL}_2(p)$ with p a Mersenne prime.

Now G acts faithfully on a nontrivial system of imprimitivity \mathcal{P} for Ω , which we may identify with the Cartesian product Δ^k . Let $A \in \mathcal{P}$ be the block containing α . Without loss of generality, we may assume that $A = (\delta, \dots, \delta)$ and $\text{soc}(L)_\delta = C_p : C_{(p-1)/2}$. In particular, $|\Delta| = |\text{soc}(L) : \text{soc}(L)_\delta| = p+1$. Since we are assuming that the orbital graph corresponding to ω^{G_α} is connected, Lemma 5.2 implies that $\omega \notin A$. Moreover, if $B = (b_1, \dots, b_k) \in \mathcal{P}$ is the block containing ω , then the same lemma also implies that $|B^{G_\alpha}|$ divides $|\omega^{G_\alpha}|$.

As in the statement of Theorem 2,

$$(C_p : C_r)^k \leq N_\alpha < (C_p : C_{(p-1)/2})^k = N_A,$$

where r is the product of the distinct prime divisors of $(p-1)/2$. Note that $O_p(N_\alpha) \leq G_\alpha$ and thus $|B^{O_p(N_\alpha)}|$ divides $|B^{G_\alpha}|$. Now $O_p(\text{soc}(L)_\delta) = C_p$ has one fixed point and one orbit of length p on Δ , hence $|B^{O_p(N_\alpha)}| = p^\ell$, where $\ell = |\{i \mid b_i \neq \delta\}|$. Since $|\omega^{G_\alpha}|$ is a prime, we deduce that $\ell = 1$ and

$$|\omega^{G_\alpha}| = |B^{G_\alpha}| \text{ and } B^{G_\alpha} = B^{O_p(N_\alpha)}.$$

In particular, each element of $B^{O_p(N_\alpha)}$ differs from A in precisely the same coordinate. Now G transitively permutes the k simple direct factors of N and we have $G = NG_\alpha$ (since N is transitive on Ω), hence G_α also acts transitively on the factors of N . Therefore, for each $i \in \{1, \dots, k\}$ there is an element of B^{G_α} that differs from A in the i -th coordinate. This is a contradiction. \square

Finally we turn our attention to $2'$ -elusive biquasiprimitive groups. Recall that a transitive group $G \leq \text{Sym}(\Omega)$ is biquasiprimitive if every nontrivial normal subgroup of G has at most two orbits on Ω , and there is a normal subgroup with two orbits, say Δ_1 and Δ_2 . Let G^+ denote the index-two subgroup of G that fixes Δ_1 and Δ_2 setwise. Also recall that if Γ is a graph with vertex set $V\Gamma$, then the *standard double cover* of Γ is the graph with vertex set $V\Gamma \times \{0, 1\}$, such that $\{(u, a), (v, b)\}$ is an edge if and only if $a \neq b$ and $\{u, v\}$ is an edge of Γ .

Lemma 5.6. *Let $G \leq \text{Sym}(\Omega)$ be a finite biquasiprimitive permutation group such that $|\Omega| > 2$ and the actions of G^+ on Δ_1 and Δ_2 are permutation isomorphic. If Γ is a connected orbital graph of G , then Γ is the standard double cover of a connected orbital graph of G^+ on Δ_1 .*

Proof. Let Γ be the connected orbital graph corresponding to a suborbit ω^{G_α} with $\alpha \in \Delta_1$. Since the actions of G^+ on Δ_1 and Δ_2 are permutation isomorphic, there exists a bijection $\varphi : \Delta_1 \rightarrow \Delta_2$ such that $\varphi(\alpha^g) = \varphi(\alpha)^g$ for all $\alpha \in \Delta_1$ and $g \in G^+$. In particular, $G_\alpha = G_{\varphi(\alpha)}$. Let $\psi : V\Gamma \rightarrow \Delta_1 \times \{0, 1\}$ be the bijection such that $\psi(\beta) = (\beta, 0)$ for each $\beta \in \Delta_1$ and $\psi(\beta) = (\varphi^{-1}(\beta), 1)$ for each $\beta \in \Delta_2$.

By Lemma 5.2, $\omega^{G_\alpha} \subseteq \Delta_2$. Moreover, since $|\Omega| > 2$ and Γ is connected, it follows that Γ is k -regular, where $k = |\omega^{G_\alpha}| \geq 2$. In particular, $\varphi(\alpha) \notin \omega^{G_\alpha}$. Let $O_1 = \varphi^{-1}(\omega^{G_\alpha}) = \varphi^{-1}(\omega)^{G_\alpha}$, so O_1 is an orbit of $G_\alpha = (G^+)_\alpha$ on $\Delta_1 \setminus \{\alpha\}$. We claim that O_1 is self-paired (as a suborbit of G^+). To see this, first observe that $\omega^{G_\alpha} = \omega^{G_{\varphi(\alpha)}}$ is self-paired, so there exists an element $g \in G$ such that $\omega^g = \varphi(\alpha)$ and $\varphi(\alpha)^g = \omega$. Therefore, g interchanges $\varphi^{-1}(\omega)$ and α , and we note that $g \in G^+$ since $\varphi(\alpha), \omega \in \Delta_2$. This justifies the claim.

Recall that G acts arc-transitively on Γ , so G^+ acts transitively on the set of arcs of Γ of the form (u, v) with $u \in \Delta_1$. Since each edge of Γ corresponds to a unique such arc, it

follows that G^+ is transitive on the set of edges of Γ . In particular, every edge of Γ is of the form $\{\alpha^g, \omega^g\}$ for some $g \in G^+$.

Set $\delta = \varphi^{-1}(\omega) \in O_1$ and let Σ be the orbital graph of G^+ on Δ_1 corresponding to O_1 . Then the edges of the standard double cover of Σ are of the form $\{(\alpha^g, 0), (\delta^g, 1)\}$ for $g \in G^+$. Since $\delta^g = \varphi^{-1}(\omega)^g = \varphi^{-1}(\omega^g)$ it follows that $\{(\alpha^g, 0), (\delta^g, 1)\}$ is the image of the edge $\{\alpha^g, \omega^g\}$ of Γ under ψ . Therefore, Γ is isomorphic to the standard double cover of Σ . Note that since Γ is connected then so is Σ . \square

Lemma 5.7. *Let $G \leq \text{Sym}(\Omega)$ be a finite $2'$ -elusive biquasiprimitive permutation group with point stabiliser H . Then G has a connected orbital graph Γ of prime valency p if and only if $(G, H) = (\text{PGL}_2(p), C_p : C_s)$, where p is a Mersenne prime, $C_r \leq C_s < C_{(p-1)/2}$ and r is the product of the distinct prime divisors of $(p-1)/2$. Moreover, Γ is the standard double cover of a connected p -regular orbital graph of $\text{PSL}_2(p)$.*

Proof. Let ω^{G_α} be a self-paired suborbit of prime length such that the corresponding orbital graph Γ is connected. Without loss of generality, we may assume that $\alpha \in \Delta_1$. Then $\omega \in \Delta_2$ by Lemma 5.2.

First assume that G^+ is not faithful on at least one of its orbits, in which case the structure of G is described in Theorem 3(c). In every case, we observe that the kernel of the action of G^+ on Δ_1 is transitive on Δ_2 (see the proof of Lemma 4.6, for example). Therefore, any connected orbital graph arising from such a group is complete and bipartite. But $|\Delta_1|$ is not a prime, so this situation does not arise.

For the remainder we may assume that G^+ is faithful on both orbits, in which case the structure of G is given in Theorem 3(b). Let $N = T^k$ be the unique minimal normal subgroup of G , where $k \geq 1$. Let $K^+ \leq S_k$ be the group induced by G^+ on the set of k simple direct factors of N . Recall that $G^+ = NG_\alpha$ (see Lemma 4.1).

If $k = 1$ then G is almost simple and thus $(G, G_\alpha) = (\text{PGL}_2(p), C_p : C_s)$ by Lemma 5.3. Here $G_\alpha < T = G^+$ and $|N_G(G_\alpha) : N_{G^+}(G_\alpha)| = 2$, so there exists an element $g \in N_G(G_\alpha) \setminus G^+$. Therefore, $(G^+)_\alpha = G_\alpha = G_{\alpha^g} = (G^+)_{\alpha^g}$ and $\alpha^g \in \Delta_2$, so the actions of G^+ on Δ_1 and Δ_2 are permutation isomorphic. Therefore, Lemma 5.6 implies that Γ is the standard double cover of a connected p -regular orbital graph of T on Δ_1 .

To complete the proof, we may assume that $k \geq 2$. Once again, the actions of G^+ on Δ_1 and Δ_2 are permutation isomorphic, so the lengths of the orbits of G_α on Δ_2 are the same as those on Δ_1 . By Lemma 5.6, Γ is the standard double cover of a connected orbital graph Σ of G^+ on Δ_1 of prime valency. If $T = M_{11}$ or ${}^2F_4(2)'$ then G^+ acts on Δ_1 with its standard product action, so by appealing to Lemma 5.4 we deduce that G does not have an appropriate self-paired suborbit. This is a contradiction.

Finally, let us assume that $k \geq 2$ and $T = \text{PSL}_2(p)$. To eliminate this case, we will show that G^+ does not have a connected orbital graph on Δ_1 of prime valency. Seeking a contradiction, suppose $\beta^{(G^+)_\alpha} = \beta^{G_\alpha} \subseteq \Delta_1$ be a self-paired suborbit of prime length with connected orbital graph Σ . Here Δ_1 admits a G^+ -invariant partition \mathcal{P} that can be identified with an appropriate Cartesian product Δ^k . Let $A \in \mathcal{P}$ be the block containing α . Without loss of generality, we may assume that $A = (\delta, \dots, \delta)$ for some $\delta \in \Delta$. Similarly, let $B \in \mathcal{P}$ be the block containing β .

By Lemma 5.2 it follows that $A \neq B$ and $|B^{G_\alpha}|$ divides $|\beta^{G_\alpha}|$. If $\beta^{G_\alpha} \subseteq B$ then the connectivity of Σ implies that $\mathcal{P} = \{A, B\}$, which is a contradiction since $|\Delta^k| > 2$. Therefore $|B^{G_\alpha}| = |\beta^{G_\alpha}|$ and by applying Lemma 5.4 we deduce that $k = 2$ and $K^+ = 1$. In particular, $G^+ \leq \text{PGL}_2(p) \times \text{PGL}_2(p)$. By arguing as in the proof of Lemma 5.4 we see that B differs from A in precisely one coordinate. Without loss of generality we may assume that $B = (\gamma, \delta)$ for some $\gamma \neq \delta$. If $g = (g_1, g_2) \in G^+$ interchanges α and β , then g also interchanges A and B and we have $\delta^{g_2} = \delta$. It follows that $\delta^{h_2} = \delta$ for each $(h_1, h_2) \in \langle G_\alpha, g \rangle \leq \langle G_A, g \rangle$. Therefore, $\langle G_\alpha, g \rangle \neq G$ and thus Lemma 5.1 implies that Σ is disconnected, a contradiction. \square

Finally, we are now ready to prove Theorem 5 and Corollaries 6 and 7.

Proof of Theorem 5. Let Γ be a finite connected graph of prime valency p and let $G \leq \text{Aut}(\Gamma)$ be an arc-transitive group of automorphisms so that the action of G on the vertex set $V\Gamma$ is either quasiprimitive or biquasiprimitive. We may assume that G is $2'$ -elusive on $V\Gamma$ (otherwise case (i) or (ii) in Theorem 5 holds). Then G is almost simple by Lemmas 5.5 and 5.7, and the result now follows by applying Lemmas 5.3 and 5.7. \square

Proof of Corollary 6. Let Γ be a finite connected graph of prime valency and let $G \leq \text{Aut}(\Gamma)$ be an elusive arc-transitive group of automorphisms. Since the valency is prime, for each vertex $v \in V\Gamma$, the action of G_v on the set of neighbours of v is primitive. Let N be a normal subgroup of G and suppose that N has at least three orbits on vertices. Then by [21, Lemma 1.6], N is semiregular, which contradicts the fact that G is elusive. Therefore, G is either quasiprimitive or biquasiprimitive on vertices, and thus G and Γ are given by Theorem 5(ii)–(v). If $|V\Gamma|$ is a power of two then [17, Proposition 3.2] implies that G contains a derangement of order two, which is a contradiction (the proof in [17] applies to any vertex-transitive subgroup, not just the full automorphism group). In cases (iv) and (v) of Theorem 5, note that $|G|$ is even and $|G_v|$ is odd, so once again we deduce that G contains a derangement of order two. Therefore, the only possibility is the example in part (iii), hence $G = M_{11}$ and Γ is the complete graph K_{12} . \square

Proof of Corollary 7. Let k be the smallest integer such that there is a finite connected graph of valency k with an elusive arc-transitive group of automorphisms. By [14, Theorem 1.1], $k \geq 5$. As discussed in the introduction to [14], the example in [14, Theorem 3.5(3)] shows that $k \leq 6$. Since Corollary 6 shows that $k = 5$ is not possible, we conclude that $k = 6$. \square

REFERENCES

- [1] B. Alspach, *Lifting Hamilton cycles of quotient graphs*, Discrete Math. **78** (1989), 25–36.
- [2] N. Biggs, *Three remarkable graphs*, Canad. J. Math. **25** (1973), 397–411.
- [3] W. Bosma, J. Cannon and C. Playoust, *The MAGMA algebra system I: The user language*, J. Symbolic Comput. **24** (1997), 235–265.
- [4] T.C. Burness and M. Giudici, *Classical groups, derangements and primes*, Aust. Math. Soc. Lecture Series, vol. 25, Cambridge University Press, 2016.
- [5] T.C. Burness, M. Giudici and R.A. Wilson, *Prime order derangements in primitive permutation groups*, J. Algebra **341** (2011), 158–178.
- [6] P.J. Cameron, M. Giudici, G.A. Jones, W.M. Kantor, M.H. Klin, D. Marušič and L.A. Nowitz, *Transitive permutation groups without semiregular subgroups*, J. London Math. Soc. **66** (2002), 325–333.
- [7] P. Cameron, J. Sheehan and P. Spiga, *Semiregular automorphisms of vertex-transitive cubic graphs*, European J. Combin. **27** (2006), 924–930.
- [8] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.
- [9] A. Devillers, M. Giudici, C.H. Li and C.E. Praeger, *Locally s -distance transitive graphs*, J. Graph Theory **69** (2012), 176–197.
- [10] J.D. Dixon and B. Mortimer, *Permutation groups*, Springer-Verlag, New York, 1996.
- [11] B. Fein, W.M. Kantor and M. Schacher, *Relative Brauer groups II*, J. Reine Angew. Math. **328** (1981), 39–57.
- [12] M. Giudici, *Quasiprimitive groups with no fixed point free elements of prime order*, J. London Math. Soc. **67** (2003), 73–84.
- [13] M. Giudici and S. Kelly, *Characterizing a family of elusive permutation groups*, J. Group Theory **12** (2009), 95–105.
- [14] M. Giudici, L. Morgan, P. Potočník and G. Verret, *Elusive groups of automorphisms of digraphs of small valency*, European J. Combin. **46** (2015), 1–9.
- [15] M. Giudici and J. Xu, *All vertex-transitive locally-quasiprimitive graphs have a semiregular automorphism*, J. Algebraic Combin. **25**, (2007), 217–232.
- [16] M.W. Liebeck, C.E. Praeger and J. Saxl, *Transitive subgroups of primitive permutation groups*, J. Algebra **234** (2000), 291–361.
- [17] D. Marušič, *On vertex symmetric digraphs*, Discrete Math. **36** (1981), 69–81.
- [18] D. Marušič and R. Scapellato, *Permutation groups, vertex-transitive digraphs and semiregular automorphisms*, European J. Combin. **19** (1998), 707–712.
- [19] B.D. McKay and G.F. Royle, *The transitive graphs with at most 26 vertices*, Ars Combin. **30** (1990), 161–176.

- [20] J. Morris, P. Spiga and G. Verret, *Semiregular automorphisms of cubic vertex-transitive graphs and the abelian normal quotient method*, Electron. J. Combin. **22**, (2015) #P3.32.
- [21] C.E. Praeger, *Imprimitive symmetric graphs*, Ars Combin. **19** (1985), 149–163.
- [22] C.E. Praeger, *An O’Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs*, J. London Math. Soc. **47** (1993), 227–239.
- [23] C.E. Praeger, *Finite transitive permutation groups and bipartite vertex-transitive graphs*, Illinois J. Math. **47** (2003), 461–475.
- [24] J.-P. Serre, *On a theorem of Jordan*, Bull. Amer. Math. Soc. **40** (2003), 429–440.
- [25] J.J. Sylvester, *On arithmetical series*, Messenger of Math. **21** (1892), 1-19, 87-120; and *The Collected Mathematical Works of James Joseph Sylvester*, vol. 4, Cambridge University Press, 1912, pp.687–731.
- [26] J. Xu, *On elusive permutation groups of square-free degree*, Comm. Algebra **37** (2009), 3200–3206.

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